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Full Length Research Paper

The exp-function method for new exact solutions of the nonlinear partial differential equations

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In this article, the exp-function method is used to construct some new exact solitary wave solutions of the sixth-order Boussinesq equation and the regularized long wave equations. These equations play very important role in mathematical physics, engineering sciences and applied mathematics. The exp-function method is a powerful and straightforward mathematical tool for solving nonlinear evolution equations.

Key words: Sixth-order Boussinesq equation, regularized long wave equation, the exp-function method, solitary wave solutions, nonlinear partial differential equations.

INTRODUCTION

In the study of nonlinear physical phenomena, searching exact traveling wave solutions play an important role. The occurrence of wave phenomena is seen in elastic media, fluid dynamics, optical fibers, plasma (Abdou et al., 2007), etc. In the present time, many new efficient and powerful methods have been suggested by different scientists to find the exact traveling wave solutions of nonlinear evolution equations (NLEEs), such as, the Hirota's bilinear method (Hirota, 1971), the Backlund transformation method (Rogers and Shadwick, 1982), the inverse scattering method (Ablowitz and Clarkson, 1991), the tanh –function method (Malfliet, 1992), the homotopy analysis method (Liao, 1992; Mohyud-Din et al., 2011; Ezzati and Aqhamohamadi, 2011), the Jacobi elliptic function expansion method (Liu et al., 2001), the F-expansion method (Zhou et al., 2003; Zhang et al., 2005), the variational iteration method (He, 1997; Yousefi et al., 2009; Jafari et al., 2011), (G'/G)– expansion method (Neirameh et al., 2010) and so on.

Recently, He and Wu (2006) presented a method, called the exp-function method to seek solitary wave solutions, periodic solutions and compacton-like solutions of nonlinear evolution equations. After introducing the method, it caught instantaneous attention and it has been widely used for searching exact traveling wave solutions for the different partial differential equations. For example, He and Abdou (2007) investigated the method for searching new periodic solutions of the nonlinear wave equations; Wu and He (2007) used the method for finding solitary solutions and periodic solutions; Zhang (2010, 2008) studied the method for constructing explicit and exact solutions of a lattice equation, whilst he (2010) implemented the method to get exact solutions for Riccati equation with arbitrary function in the another article; Bekir and Aksoy (2010) concerned the method for NLEEs with variable coefficients; Yildirim and Pinar (2010) applied the method for reaction-diffusion equations; Asgari et al. (2011) implemented the method for infiltration equation; Naher et al. (2011) investigated the higher dimensional nonlinear partial differential equation by the method; Misirli and Gurefe (2011) studied the method for solving NLEEs and so on. The study of the exp-function method indicates that it is a very effective and trustworthy method.

In this article, we use the exp-function method to obtain new solitary wave solutions for the sixth-order Boussinesq equation and the regularized long wave equations.

Description of the exp-function method

Suppose the general nonlinear partial differential equation in two independent variables x and t, is:

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$$P\left(u,u_t,u_x,u_{tt},u_{xt},u_{xx},...\right)=0, \quad (1)$$

where $u=u\left(x,t\right)$ is an unknown function and P is a polynomial in $u=u\left(x,t\right)$ which has various partial derivatives, and the highest order derivatives and nonlinear terms are involved in it. The important steps of the exp-function method are discussed in the following:

Consider the traveling wave transformation:

$$u\left(x,t\right)=u\left(\xi\right), \qquad \xi=x+st, \quad (2)$$

where s is the wave speed and ξ is the combination of two independent variables x and t. Using Equation (2), Equation (1) transforms to an ordinary differential equation:

$$Q\left(u,su',u',s^2u'',su'',u'',...\right)=0, \quad (3)$$

where primes denote the ordinary derivative with respect to ξ.

We assume that the wave solution of Equation (3) can be expressed in the form (He and Wu, 2006):

$$u\left(\xi\right)=\frac{\sum_{n=-c}^{d}a_n\exp\left(n\xi\right)}{\sum_{m=-p}^{q}b_m\exp\left(m\xi\right)}=\frac{a_{-c}\exp\left(-c\xi\right)+...+a_d\exp\left(d\xi\right)}{b_{-p}\exp\left(-p\xi\right)+...+b_q\exp\left(q\xi\right)}, \quad (4)$$

where c,d,p and q are positive integers that could be determine subsequently, a_n and b_m are unknown constants , Equation (4) can be re-written in the form:

$$u\left(\xi\right)=\frac{a_c\exp\left(c\xi\right)+...+a_{-d}\exp\left(-d\xi\right)}{b_p\exp\left(p\xi\right)+...+b_{-q}\exp\left(-q\xi\right)}. \quad (5)$$

In order to determine the values of c and p, we balance the highest order linear term with the highest order nonlinear term in Equation (3).

Similarly, to determine the values of d and q, we balance the lowest order linear term with the lowest order nonlinear term in Equation (3).

Putting the values of c, d, p and q into Equation (5) and then substituting Equation (5) into Equation (3) and simplifying, we obtain;

$$\sum_k C_k\exp\left(\pm k\xi\right)=0, \quad k=0,1.2,3,... \quad (6)$$

Then setting each coefficient $C_k=0,$ yields a set of algebraic equations for a_c's and b_p's. The unknown a_c's and b_p's can be obtained by solving the algebraic equations. Substituting these values into Equation (5), we obtain traveling wave solutions of the Equation (1).

APPLICATIONS

Here, we apply the exp-function method for obtaining some new exact solitary wave solutions of two important NLEEs, namely, the sixth-order Boussinesq equation and the regularized long wave equations. The explanation between obtained solutions and solutions obtained in literature is discussed. Moreover, the solutions are shown in some graphs with the aid of algebraic software Maple 13.

Solutions of the sixth-order Boussinesq equation

Consider the sixth-order Boussinesq equation (Aslan and Ozis, 2009):

$$u_{tt}-u_{xx}-\left(15uu_{4x}+30u_xu_{3x}+15\left(u_{2x}\right)^2+45u^2u_{2x}+90uu_x^2+u_{6x}\right)=0, \quad (7)$$

Now, we seek traveling wave solutions of Equation (7). We use Equation (2) into Equation (7), which yields:

$$s^2u''-u''-\left(15uu^{(4)}+30u'u'''+15\left(u''\right)^2+45u^2u''+90u\left(u'\right)^2+u^{(6)}\right)=0. \quad (8)$$

where primes denote the derivative with respect to ξ. According to He and Wu (2006), the solution of Equation (8) can be expressed in the form of Equation (5).

To determine the values of c and p, we balance the highest order linear term of $u^{(6)}$ with the highest order nonlinear term of $uu^{(4)}$ in Equation (8). Therefore, we have

$$u^{(6)}=\frac{c_1\exp\left[\left(6p+c\right)\xi\right]+...}{c_2\exp\left[7p\xi\right]+...}, \quad (9)$$

and

$$uu^{(4)}=\frac{c_3\exp\left[\left(5p+2c\right)\xi\right]+...}{c_4\exp\left[7p\xi\right]+...}, \quad (10)$$

where c_i are coefficients for simplicity. By balancing the highest order of the exp-function in Equations (9) and (10), we obtain $6p+c=5p+2c$, which in turn gives $p=c$.

Again, to determine the values of d and q, we balance the lowest order linear term of $u^{(6)}$ with the lowest order nonlinear term of $uu^{(4)}$ in Equation (8). Therefore, we have

$$u^{(6)}=\frac{...+d_1\exp\left[-(d+6q)\xi\right]}{...+d_2\exp\left[-7q\xi\right]}, \qquad (11)$$

and

$$uu^{(4)}=\frac{...+d_3\exp\left[-(2d+5q)\xi\right]}{...+d_4\exp\left[-7q\xi\right]}, \qquad (12)$$

where d_j are determined coefficients only for simplicity. By balancing the lowest order of the exp-function in Equations (11) and (12), we obtain $-(d+6q)=-(2d+5q)$, which in turn gives $q=d$.

Here, we can freely choose the values of c and d, but the final solution does not depend upon the choice of the values of c and d.

Case 1

Choose $p=c=1$ and $q=d=1$.
For this case, the trial solution Equation (5) reduces to

$$u(\xi)=\frac{a_1e^{\xi}+a_0+a_{-1}e^{-\xi}}{b_1e^{\xi}+b_0+b_{-1}e^{-\xi}}. \qquad (13)$$

In case $b_1\neq 0$, Equation (13) can be simplified as:

$$u(\xi)=\frac{a_1e^{\xi}+a_0+a_{-1}e^{-\xi}}{e^{\xi}+b_0+b_{-1}e^{-\xi}}. \qquad (14)$$

Now substituting Equation (14) into Equation (8) and simplifying, we have

$$\frac{1}{A}(C_6e^{6\xi}+C_5e^{5\xi}+C_4e^{4\xi}+C_3e^{3\xi}+C_2e^{2\xi}+C_1e^{\xi}+C_0+C_{-1}e^{-\xi}+C_{-2}e^{-2\xi}+C_{-3}e^{-3\xi}+C_{-4}e^{-4\xi}+C_{-5}e^{-5\xi}+C_{-6}e^{-6\xi})=0, \qquad (15)$$

where $A=\left(e^{\xi}+b_0+b_{-1}e^{-\xi}\right)^7$.

Setting each coefficient of $\exp(\pm n\xi)$, $n=0,1,2,3,...,$ we obtain a set of algebraic equations for a_{-1}, a_0, a_1, b_{-1}, b_0 and s.

$$\begin{aligned}
C_6 &= -2a_0 - s^2a_1b_0 + s^2a_0 + 2a_1b_0 - 45a_0a_1^2 + 45a_1^3b_0 + 15a_1^2b_0 - 15a_1a_0 = 0,\\
C_5 &= 3s^2a_0b_0 + 255a_0a_1b_0 - 60a_0^2 - 54a_1b_0^2 - 68a_{-1} + 54a_0b_0 - 4s^2a_1b_{-1} - 3s^2a_1b_0^2 - 45a_1^3b_0^2 - 180a_1a_0^2\\
&\quad + 225a_1^2a_0b_0 + 68a_1b_{-1} + 4s^2a_{-1} + 180a_1^3b_{-1} - 180a_1^2a_{-1} + 240a_1^2b_{-1} - 240a_1a_{-1} - 195a_1^2b_0^2 = 0,\\
C_4 &= 262a_{-1}b_0 + 724a_0b_{-1} - 986a_1b_0b_{-1} - 135a_0^3 + 900a_1^2a_0b_{-1} - 1320a_1^2b_0b_{-1} + 210a_1b_0a_{-1} + 1785a_1b_{-1}a_0\\
&\quad - 45a_1^3b_0b_{-1} - 2s^2a_0b_{-1} + 19s^2a_{-1}b_0 - 360a_0a_1b_0^2 - 2s^2a_1b_0^3 + 2s^2a_0b_0^2 + 405a_1^2a_0b_0^2 - 45a_1a_0^2b_0\\
&\quad - 45a_1^2b_0a_{-1} - 810a_1a_{-1}a_0 - 17s^2a_1b_0b_{-1} + 304a_1b_0^3 + 255a_0^2b_0 - 675a_0a_{-1} + 105a_1^2b_0^3 - 304a_0b_0^2\\
&\quad - 225a_1^3b_0^3 = 0,\\
C_3 &= -960a_{-1}^2 + 900a_1b_{-1}a_0^2 - 720a_{-1}a_0^2 - 720a_1a_{-1}^2 + 30a_1b_0^2a_{-1} + 4080a_1b_{-1}a_{-1} - 990a_1^3b_0^2b_{-1} - 405a_{-1}a_0b_0\\
&\quad - 2666a_0b_0b_{-1} + 3040a_1b_0^2b_{-1} + 75a_1^2b_0^2b_{-1} - 12s^2a_1b_{-1}^2 + 12s^2a_{-1}b_{-1} + 37s^2a_{-1}b_0^2 - 360a_0a_1b_0^3 + 2s^2a_1b_0^4\\
&\quad - 2s^2a_0b_0^3 + 405a_1a_0^2b_0^2 - 45a_1^2a_0b_0^3 + 405a_1^2b_0^2a_{-1} + 900a_1^2b_{-1}a_{-1} - 17s^2a_0b_0b_{-1} + 1755a_1^2a_0b_0b_{-1}\\
&\quad - 1350a_1a_0a_{-1}b_0 - 1680a_0a_1b_0b_{-1} - 20s^2a_1b_0^2b_{-1} - 304a_1b_0^4 - 3120a_1^2b_{-1}^2 + 1980a_0^2b_{-1} + 105a_0^2b_0^2\\
&\quad + 255a_1^2b_0^4 + 304a_0b_0^3 - 135a_1^3b_0^4 - 225a_0^3b_0 - 180a_1^3b_{-1}^2 - 3636a_1b_{-1}^2 + 3636a_{-1}b_{-1} - 374a_{-1}b_0^2 = 0,
\end{aligned}$$

$$
\begin{aligned}
C_2 = {} & -10526a_0b_{-1}^2 - 45a_0^3b_0^2 - 1665a_0^2a_{-1}b_0 - 1125a_0a_{-1}^2 - 206a_{-1}b_0b_{-1} - 1665a_1^3b_0b_{-1}^2 - 765a_1^3b_0^3b_{-1} \\
& - 7710a_1b_{-1}^2a_0 + 810a_1^2a_0b_{-1}^2 + 6525a_{-1}a_0b_{-1} - 60a_0a_{-1}b_0^2 - 330a_1b_0^3a_{-1} + 2360a_0b_0^2b_{-1} + 10732a_1b_0b_{-1}^2 \\
& - 2344a_1b_0^3b_{-1} + 1380a_0^2b_0b_{-1} + 1830a_1^2b_0^3b_{-1} - 690a_1^2b_{-1}^2b_0 - 17s^2a_0b_{-1}^2 + 38s^2a_{-1}b_0^3 + 255a_1b_0^4a_0 \\
& + 3s^2a_1b_0^5 - 3s^2a_0b_0^4 - 180a_1^2a_0b_0^4 + 225a_1a_0^2b_0^3 + 225a_1^2b_0^3a_{-1} - 1665a_1a_{-1}^2b_0 + 1080a_1a_0a_{-1}b_{-1} \\
& - 2820a_0b_{-1}b_0^2a_1 + 3720a_1b_0a_{-1}b_{-1} + 2565a_1a_0^2b_0b_{-1} + 2565a_1^2a_{-1}b_0b_{-1} - 270a_1^2a_0b_{-1}b_0^2 - 270a_1a_0a_{-1}b_0^2 \\
& - 40s^2a_0b_0^2b_{-1} - 26s^2a_1b_0b_{-1}^2 + 2s^2a_1b_0^3b_{-1} + 43s^2a_{-1}b_0b_{-1} + 54a_1b_0^5 - 1845a_{-1}^2b_0 - 60a_1^2b_0^5 - 195a_0^2b_0^3 \\
& - 54a_0b_0^4 - 16a_{-1}b_0^3 + 180a_0^3b_{-1} = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_1 = {} & 4590a_0a_1b_{-1}a_{-1}b_0 - 540a_{-1}^3 - 10290a_1b_{-1}^2a_0b_0 + 10440a_{-1}b_0a_0b_{-1} - 390a_{-1}b_0^2a_1b_{-1} + 1740a_0b_{-1}a_1b_0^3 \\
& + 1350a_1^2b_{-1}a_{-1}b_0^2 + 1350a_1b_{-1}a_0^2b_0^2 - 1485a_0a_1^2b_{-1}^2b_0 - 1125a_0a_1^2b_0^3b_{-1} + 270a_0a_1b_0^3a_{-1} - 66s^2a_0b_0b_{-1}^2 \\
& - 38s^2a_0b_0^3b_{-1} + 16s^2a_1b_0^4b_{-1} + 6s^2a_1b_0^2b_{-1}^2 + 60s^2a_{-1}b_0^2b_{-1} - 2a_1b_0^6 - 1695a_{-1}^2b_0^2 + 45a_0^3b_0^3 + 2a_0b_0^5 + 15a_0^2b_0^4 \\
& + 19336a_1b_{-1}^3 - 19336a_{-1}b_{-1}^2 - 44a_{-1}b_0^4 + 4080a_{-1}^2b_{-1} + 1680a_1^2b_{-1}^3 - 900a_1^3b_{-1}^3 - 3360a_0^2b_{-1}^2 + 5295a_1^2b_0^2b_{-1}^2 \\
& + 90a_{-1}b_0^4a_1 - 420a_{-1}b_0^3a_0 - 1005a_0^2b_{-1}b_0^2 - 405a_1^2b_0^4b_{-1} - s^2a_0b_0^5 - 45a_0^2a_1b_0^4 - 15a_0a_1b_0^5 - 3015a_0a_{-1}^2b_0 \\
& - 1665a_1^3b_{-1}^2b_0^2 - 1215a_1a_{-1}^2b_0^2 - 1215a_{-1}a_0^2b_0^2 - 5760a_1b_{-1}^2a_{-1} + 5172a_0b_0b_{-1}^2 - 344a_0b_0^3b_{-1} + 388a_1b_0^4b_{-1} \\
& - 7572a_1b_0^2b_{-1}^2 + 2400a_{-1}b_0^2b_{-1} - 8s^2a_1b_{-1}^3 + 8s^2a_{-1}b_{-1}^2 + 22s^2a_{-1}b_0^4 + 1620a_1a_0^2b_{-1}^2 + 765a_0^3b_0b_{-1} \\
& - 180a_1a_{-1}^2b_{-1} - 180a_0^2a_{-1}b_{-1} + 1620a_1^2a_{-1}b_{-1}^2 - 45a_1^2a_{-1}b_0^4 + s^2a_1b_0^6 = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_0 = {} & -1155a_{-1}^2b_0^3 - 7350a_0^2b_{-1}^2b_0 - 1155a_1^2b_0^3b_{-1}^2 + 4200a_{-1}b_0^2a_0b_{-1} + 1260a_{-1}b_0^3a_1b_{-1} + 4200a_0b_{-1}^2a_1b_0^2 \\
& - 2835a_0a_1^2b_{-1}^2b_0^2 - 2835a_0a_{-1}^2b_0^2 + 3780a_0a_1b_{-1}a_{-1}b_0^2 + 945a_1a_{-1}^2b_0b_{-1} + 945a_1a_0^2b_{-1}^2b_0 - 315a_1a_0^2b_{-1}b_0^3 \\
& + 3780a_1a_0a_{-1}b_{-1}^2 - 105a_0b_{-1}b_0^4a_1 - 12180a_1b_0a_{-1}b_{-1}^2 + 945a_1^2a_{-1}b_0b_{-1}^2 - 315a_1^2a_{-1}b_0^3b_{-1} - 84s^2a_0b_0^2b_{-1}^2 \\
& - 14s^2a_0b_{-1}b_0^4 + 7s^2a_1b_0^5b_{-1} + 42s^2a_1b_0^3b_{-1}^2 + 14s^2a_1b_0b_{-1}^3 + 14s^2a_{-1}b_0b_{-1}^2 + 42s^2a_{-1}b_0^3b_{-1} - 2310a_{-1}a_0b_{-1}^2 \\
& - 2310a_1b_{-1}^3a_0 - 105a_{-1}b_0^4a_0 - 2352a_0b_0^2b_{-1}^2 + 28a_0b_{-1}b_0^4 - 14a_1b_0^5b_{-1} + 1176a_1b_0^3b_{-1}^2 - 11788a_1b_0b_{-1}^3 \\
& - 11788a_{-1}b_0b_{-1}^2 + 1176a_{-1}b_0^3b_{-1} + 8400a_{-1}^2b_0b_{-1} + 8400a_1^2b_{-1}^3b_0 + 210a_0^2b_{-1}b_0^3 - 28s^2a_0b_{-1}^3 + 23576a_0b_{-1}^3 \\
& - 14a_{-1}b_0^5 + 7s^2a_{-1}b_0^5 + 630a_0^3b_{-1}^2 - 1575a_{-1}^3b_0 + 945a_0^2a_{-1}b_0b_{-1} + 630a_0^3b_0^2b_{-1} - 1260a_0a_{-1}^2b_{-1} - 315a_0^2a_{-1}b_0^3 \\
& - 1575a_1^3b_{-1}^3b_0 - 1260a_1^2a_0b_{-1}^3 - 315a_1a_{-1}^2b_0^3 = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_{-1} = {} & -19336a_1b_{-1}^4 + 8s^2a_1b_{-1}^4 - 8s^2a_{-1}b_{-1}^3 + 765a_0^3b_{-1}^2b_0 + 1620a_0^2a_{-1}b_{-1}^2 + 1620a_1a_{-1}^2b_{-1}^2 - 180a_1^2a_{-1}b_{-1}^3 \\
& - 180a_1a_0^2b_{-1}^3 - 45a_1a_{-1}^2b_0^4 + s^2a_{-1}b_0^6 + 4590a_0a_1b_{-1}^2a_{-1}b_0 + 270a_0a_1b_{-1}a_{-1}b_0^3 + 5295a_{-1}^2b_0^2b_{-1} \\
& - 1695a_1^2b_0^2b_{-1}^3 - 1005a_0^2b_{-1}^2b_0^2 + 2a_0b_{-1}b_0^5 + 45a_0^3b_{-1}b_0^3 - 45a_0^2a_{-1}b_0^4 - 15a_0a_{-1}b_0^5 + 15a_0^2b_{-1}b_0^4 \\
& - 1125a_0a_{-1}^2b_0^3 - 5760a_1b_{-1}^3a_{-1} + 5172a_0b_0b_{-1}^3 - 344a_0b_{-1}^2b_0^3 - 44a_1b_0^4b_{-1}^2 + 2400a_1b_0^2b_{-1}^3 + 388a_{-1}b_0^4b_{-1} \\
& - 7572a_{-1}b_0^2b_{-1}^2 - s^2a_0b_{-1}b_0^5 - 420a_0b_{-1}^2b_0^3a_1 + 90a_{-1}b_0^4a_1b_{-1} + 10440a_1b_{-1}^3a_0b_0 - 10290a_{-1}b_0a_0b_{-1}^2 \\
& - 390a_{-1}b_0^2a_1b_{-1}^2 + 1740a_{-1}b_0^3a_0b_{-1} - 1485a_0a_{-1}^2b_0b_{-1} - 1215a_1b_{-1}^2a_0^2b_0^2 + 1350a_{-1}a_0^2b_{-1}b_0^2 - 1215a_{-1}a_1^2b_{-1}^2b_0^2 \\
& + 1350a_{-1}^2a_1b_{-1}b_0^2 - 3015a_0a_1^2b_{-1}^3b_0 - 66s^2a_0b_0b_{-1}^3 - 38s^2a_0b_{-1}^2b_0^3 + 22s^2a_1b_0^4b_{-1}^2 + 60s^2a_1b_0^2b_{-1}^3 + 16s^2a_{-1}b_0^4b_{-1} \\
& + 6s^2a_{-1}b_0^2b_{-1}^2 - 3360a_0^2b_{-1}^3 - 2a_{-1}b_0^6 - 405a_{-1}^2b_0^4 - 1665a_{-1}^3b_0^2 + 19336a_{-1}b_{-1}^3 + 4080a_1^2b_{-1}^4 + 1680a_{-1}^2b_{-1}^2 \\
& - 540a_1^3b_{-1}^4 - 900a_{-1}^3b_{-1} = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_{-2} = {} & 180a_0^3b_{-1}^3 - 270a_1a_0a_{-1}b_0^2b_{-1}^2 - 10526a_0b_{-1}^4 - 7710a_{-1}a_0b_{-1}^3 + 6525a_1b_{-1}^4a_0 - 690a_{-1}^2b_{-1}^2b_0 + 2360a_0b_{-1}^3b_0^2 \\
& - 16a_1b_0^3b_{-1}^3 - 206a_1b_0b_{-1}^4 - 2344a_{-1}b_0^3b_{-1}^2 + 10732a_{-1}b_0b_{-1}^3 + 1830a_{-1}^2b_0^3b_{-1} + 1380a_0^2b_{-1}^3b_0 - 1845a_1^2b_{-1}^4b_0 \\
& - 17s^2a_0b_{-1}^4 + 54a_{-1}b_0^5b_{-1} - 54a_0b_{-1}^2b_0^4 - 45a_0^3b_{-1}^2b_0^2 - 180a_0a_{-1}^2b_0^4 - 1665a_{-1}^3b_0b_{-1} - 195a_0^2b_{-1}^2b_0^3 \\
& - 270a_0a_{-1}^2b_0^2b_{-1} + 2565a_0^2a_{-1}b_{-1}^2b_0 + 225a_0^2a_{-1}b_0^3b_{-1} - 40s^2a_0b_{-1}^3b_0^2 + 38s^2a_1b_0^3b_{-1}^3 + 43s^2a_1b_0b_{-1}^4 \\
& + 2s^2a_{-1}b_0^3b_{-1}^2 - 26s^2a_{-1}b_0b_{-1}^3 - 330a_{-1}b_0^3b_{-1}^2a_1 - 2820a_{-1}b_0^2b_{-1}^2a_0 + 3s^2a_{-1}b_0^5b_{-1} - 3s^2a_0b_{-1}^2b_0^4 + 255a_0b_{-1}b_0^4a_{-1} \\
& + 255a_1a_{-1}^2b_0^3b_{-1} + 2565a_1a_{-1}^2b_0b_{-1}^2 - 1665a_1a_0^2b_{-1}^3b_0 + 1080a_1a_0a_{-1}b_{-1}^3 + 3720a_1b_0a_{-1}b_{-1}^3 - 1665a_1^2a_{-1}b_0b_{-1}^3 \\
& - 60a_1b_0^2a_0b_{-1}^3 + 810a_0a_{-1}^2b_{-1}^2 - 60a_{-1}^2b_0^5 - 765a_{-1}^3b_0^3 - 1125a_0a_1^2b_{-1}^4 = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_{-3} = {} & 900a_{-1}a_0^2b_{-1}^3 + 3040a_{-1}b_0^2b_{-1}^3 + 4080a_1b_{-1}^4a_{-1} - 2666a_0b_{-1}^4b_0 - 374a_1b_0^2b_{-1}^4 + 75a_{-1}^2b_0^2b_{-1}^2 + 12s^2a_1b_{-1}^5 \\
& - 12s^2a_{-1}b_{-1}^4 - 304a_{-1}b_0^4b_{-1}^2 + 255a_{-1}^2b_0^4b_{-1} + 304a_0b_{-1}^3b_0^3 - 225a_0^3b_{-1}^3b_0 - 720a_{-1}a_1^2b_{-1}^4 + 900a_{-1}^2a_1b_{-1}^3 \\
& + 105a_0^2b_{-1}^3b_0^2 - 990a_{-1}^3b_0^2b_{-1} + 30a_1b_0^2a_{-1}b_{-1}^3 - 405a_1b_0a_0b_{-1}^4 - 1680a_{-1}b_0b_{-1}^3a_0 - 17s^2a_0b_{-1}^4b_0 + 37s^2a_1b_0^2b_{-1}^4 \\
& - 20s^2a_{-1}b_0^2b_{-1}^3 + 2s^2a_{-1}b_0^4b_{-1}^2 - 2s^2a_0b_{-1}^3b_0^3 - 360a_{-1}b_0^3b_{-1}^2a_0 + 405a_1a_{-1}^2b_0^2b_{-1}^2 + 1755a_0a_{-1}^2b_0b_{-1}^2 \\
& - 45a_0a_{-1}^2b_0^3b_{-1} + 405a_0^2a_{-1}b_{-1}^2b_0^2 - 960a_1^2b_{-1}^5 - 3120a_{-1}^2b_{-1}^3 + 1980a_0^2b_{-1}^4 - 135a_{-1}^3b_0^4 - 180a_{-1}^3b_{-1}^2 + 3636a_1b_{-1}^5 \\
& - 3636a_{-1}b_{-1}^4 - 720a_1b_{-1}^4a_0^2 - 1350a_1a_{-1}a_0b_{-1}^3b_0 = 0,
\end{aligned}
$$

$$
\begin{aligned}
C_{-4} = & -135a_0^3b_{-1}^4 + 210a_1b_{-1}^4a_{-1}b_0 + 19s^2a_1b_0b_{-1}^5 - 17s^2a_{-1}b_0b_{-1}^4 - 2s^2a_{-1}b_0^3b_{-1}^3 + 2s^2a_0b_{-1}^4b_0^2 - 360a_{-1}b_0^2b_{-1}^3a_0 \\
& - 45a_1a_{-1}^2b_0b_{-1}^3 - 810a_1a_{-1}a_0b_{-1}^4 + 405a_0a_{-1}^2b_0^2b_{-1}^2 - 45a_0^2a_{-1}b_{-1}^3b_0 + 724a_0b_{-1}^5 + 1785a_{-1}a_0b_{-1}^4 - 675a_1b_{-1}^5a_0 \\
& - 1320a_{-1}^2b_0b_{-1}^3 + 262a_1b_0b_{-1}^5 - 986a_{-1}b_0b_{-1}^4 - 2s^2a_0b_{-1}^5 + 304a_{-1}b_0^3b_{-1}^3 + 255a_0^2b_{-1}^4b_0 - 304a_0b_{-1}^4b_0^2 \\
& - 225a_{-1}^3b_0^3b_{-1} + 900a_{-1}^2a_0b_{-1}^3 + 105a_{-1}^2b_0^3b_{-1}^2 - 45a_{-1}^3b_0b_{-1}^2 = 0, \\
C_{-5} = & -180a_{-1}^2a_1b_{-1}^4 + 68a_{-1}b_{-1}^5 - 240a_{-1}a_1b_{-1}^5 + 4s^2a_1b_{-1}^6 - 54a_{-1}b_0^2b_{-1}^4 - 68a_1b_{-1}^6 - 45a_{-1}^3b_0^2b_{-1}^2 - 3s^2a_{-1}b_0^2b_{-1}^4 \\
& - 4s^2a_{-1}b_{-1}^5 - 195a_{-1}^2b_0^2b_{-1}^3 + 180a_{-1}^3b_{-1}^3 + 3s^2a_0b_{-1}^5b_0 + 54a_0b_{-1}^5b_0 + 240a_{-1}^2b_{-1}^4 + 255a_{-1}b_0b_{-1}^4a_0 - 60a_0^2b_{-1}^5 \\
& + 225a_0a_{-1}^2b_0b_{-1}^3 - 180a_0^2a_{-1}b_{-1}^4 = 0, \\
C_{-6} = & \, 2a_{-1}b_0b_{-1}^5 - 2a_0b_{-1}^6 - s^2a_{-1}b_0b_{-1}^5 + 15a_{-1}^2b_0b_{-1}^4 - 15a_{-1}a_0b_{-1}^5 + s^2a_0b_{-1}^6 + 45a_{-1}^3b_0b_{-1}^3 - 45a_{-1}^2a_0b_{-1}^4 = 0.
\end{aligned}
$$

Solving the system of the aforementioned algebraic equations with the aid of Maple 13, we obtain

$$b_0=b_0,\quad a_{-1}=\frac{-1}{12}b_0^2,\quad a_0=\frac{5}{3}b_0,\quad a_1=\frac{-1}{3},\quad b_{-1}=\frac{1}{4}b_0^2,\quad s=\pm\sqrt{2}, \tag{16}$$

where b_0 is arbitrary constant.

$$a_1=a_1,\quad b_0=b_0,\quad a_0=b_0(a_1+1),\quad a_{-1}=\frac{1}{4}a_1b_0^2,\quad b_{-1}=\frac{1}{4}b_0^2,\quad s=\pm\sqrt{45a_1^2+15a_1+2}, \tag{17}$$

where a_1 and b_0 are arbitrary constants.

$$b_0=b_0,\quad a_{-1}=\frac{-1}{6}b_0^2,\quad a_0=\frac{1}{3}b_0,\quad a_1=\frac{-2}{3},\quad b_{-1}=\frac{1}{4}b_0^2,\quad s=\pm 2\sqrt{3}, \tag{18}$$

where b_0 is arbitrary constant.

Now substituting Equation (16) into Equation (14), we obtain traveling wave solution:

$$u(\xi)=\frac{-4e^{\xi}+20b_0-b_0^2e^{-\xi}}{12e^{\xi}+12b_0+3b_0^2e^{-\xi}}, \tag{19}$$

Equation (19) can be simplified:

$$u(\xi)=\frac{-1}{3}+\frac{8b_0}{(4+b_0^2)\cosh\xi+(4-b_0^2)\sinh\xi+4b_0}, \tag{20}$$

where $\xi = x \pm \sqrt{2}\, t.$

If $b_0=2,$ Equation (20) becomes

$$u(x,t)=\frac{-1}{3}+\frac{2}{1+\cosh\left(x\pm\sqrt{2}\,t\right)}. \tag{21}$$

Substituting Equation (17) into Equation (14) and simplifying, we obtain

$$u(\xi)=a_1+\frac{4b_0}{(4+b_0^2)\cosh\xi+(4-b_0^2)\sinh\xi+4b_0}, \tag{22}$$

where $\xi = x \pm \sqrt{45a_1^2+15a_1+2}\; t.$

If $b_0=2,$ Equation (22) becomes

$$u(x,t)=a_1+\frac{1}{1+\cosh\left(x\pm\sqrt{45a_1^2+15a_1+2}\;t\right)}. \tag{23}$$

Again, substituting Equation (18) into Equation (14) and simplifying, we obtain

$$u(\xi)=\frac{-2}{3}+\frac{4b_0}{(4+b_0^2)\cosh\xi+(4-b_0^2)\sinh\xi+4b_0}, \tag{24}$$

where $\xi = x \pm 2\sqrt{3}\, t.$

If $b_0=2,$ Equation (24) becomes

$$u(x,t)=\frac{-2}{3}+\frac{1}{1+\cosh\left(x\pm 2\sqrt{3}\,t\right)}.$$

Case 2

Choose $p=c=2$ and $q=d=1.$

For this case, the trial solution Equation (5) reduces to

$$u(\xi)=\frac{a_2e^{2\xi}+a_1e^{\xi}+a_0+a_{-1}e^{-\xi}}{b_2e^{2\xi}+b_1e^{\xi}+b_0+b_{-1}e^{-\xi}}.$$

Since, there are some free parameters in Equation (26) for simplicity, we may consider $b_2=1$ and $b_{-1}=0.$ Then the solution Equation (26) is simplified:

$$u(\xi)=\frac{a_2e^{2\xi}+a_1e^{\xi}+a_0+a_{-1}e^{-\xi}}{e^{2\xi}+b_1e^{\xi}+b_0}. \tag{27}$$

Executing the same procedure as described in Case 1, we obtain

$$b_1=b_1,\ a_{-1}=0,\ a_0=\frac{-1}{12}b_1^2,\ a_1=\frac{5}{3}b_1,\ a_2=\frac{-1}{3},\ b_0=\frac{1}{4}b_1^2,\ s=\pm\sqrt{2},\tag{28}$$

where b1 is a free parameter

$$b_1=b_1,\ a_{-1}=0,\ a_0=\frac{-1}{12}b_1^2,\ a_1=\frac{2}{3}b_1,\ a_2=\frac{-1}{3},\ b_0=\frac{1}{4}b_1^2,\ s=\pm\sqrt{2},\tag{29}$$

where b_1 is free parameter.

$$a_0=a_0,\quad b_1=b_1,\quad a_{-1}=0,\quad a_1=\frac{4a_0+b_1^2}{b_1},\quad a_2=\frac{4a_0}{b_1^2},\quad b_0=\frac{1}{4}b_1^2,$$
$$s=\pm\frac{1}{b_1^2}\sqrt{2b_1^4+60a_0b_1^2+720a_0^2},\tag{30}$$

where a_0 and b_1 are free parameters.

$$b_1=b_1,\quad a_{-1}=0,\quad a_0=\frac{-1}{6}b_1^2,\quad a_1=\frac{1}{3}b_1,\quad a_2=\frac{-2}{3},\quad b_0=\frac{1}{4}b_1^2,\quad s=\pm 2\sqrt{3},\tag{31}$$

where b_1 is free parameter.

Using Equation (28) into Equation (27) and simplifying, we obtain

$$u(\xi)=\frac{-1}{3}+\frac{8b_1}{\left(4+b_1^2\right)\cosh\xi+\left(4-b_1^2\right)\sinh\xi+4b_1},\tag{32}$$

where $\xi=x\pm\sqrt{2}\,t.$

If $b_1=2,$ Equation (32) becomes

$$u(x,t)=\frac{-1}{3}+\frac{2}{1+\cosh\left(x\pm\sqrt{2}\,t\right)}.\tag{33}$$

Substituting Equation (29) into Equation (27) and simplifying, we obtain

$$u(\xi)=\frac{-1}{3}+\frac{4b_1}{\left(4+b_1^2\right)\cosh\xi+\left(4-b_1^2\right)\sinh\xi+4b_1},\tag{34}$$

where $\xi=x\pm\sqrt{2}\,t.$

If $b_1=2,$ Eq. (34) becomes

$$u(x,t)=\frac{-1}{3}+\frac{1}{1+\cosh\left(x\pm\sqrt{2}\,t\right)}.\tag{35}$$

Substituting Equation (30) into Equation (27) and simplifying, we obtain

$$u(\xi)=\frac{4a_0}{b_1^2}+\frac{4b_1}{\left(4+b_1^2\right)\cosh\xi+\left(4-b_1^2\right)\sinh\xi+4b_1},\tag{36}$$

where $\xi=x\pm\frac{1}{b_1^2}\sqrt{2b_1^4+60a_0b_1^2+720a_0^2}\ t.$

If $b_1=2,$ Equation (36) becomes

$$u(x,t)=a_0+\frac{1}{1+\cosh\left(x\pm\sqrt{45a_0^2+15a_0+2}\ t\right)}.\tag{37}$$

Substituting Equation (31) into Equation (27) and simplifying, we obtain

$$u(\xi)=\frac{-2}{3}+\frac{4b_1}{\left(4+b_1^2\right)\cosh\xi+\left(4-b_1^2\right)\sinh\xi+4b_1},\tag{38}$$

where $\xi=x\pm 2\sqrt{3}\,t.$

If $b_1=2,$ Equation (38) becomes

$$u(x,t)=\frac{-2}{3}+\frac{1}{1+\cosh\left(x\pm 2\sqrt{3}\,t\right)}.\tag{39}$$

Case 3

Choose $p=c=2$ and $q=d=2.$

For this case, the trial solution Equation (5) reduces to

$$u(\xi)=\frac{a_2e^{2\xi}+a_1e^{\xi}+a_0+a_{-1}e^{-\xi}+a_{-2}e^{-2\xi}}{b_2e^{2\xi}+b_1e^{\xi}+b_0+b_{-1}e^{-\xi}+b_{-2}e^{-2\xi}}.\tag{40}$$

Since, there are some free parameters in Equation (40), we may consider $b_2=1,$ $a_{-2}=0,$ $b_{-2}=0$ and $b_{-1}=0.$ So that the Equation (40) reduces to the Equation (27). This indicates that the Case 3 is equivalent to the Case 2.

Equation (40) can be re-written as:

$$u(\xi)=\frac{a_2e^{\xi}+a_1+a_0e^{-\xi}+a_{-1}e^{-2\xi}+a_{-2}e^{-3\xi}}{b_2e^{\xi}+b_1+b_0e^{-\xi}+b_{-1}e^{-2\xi}+b_{-2}e^{-3\xi}},\tag{41}$$

If we put $a_{-2}=0,$ $a_{-1}=0,$ $b_2=1,$ $b_{-2}=0$ and $b_{-1}=0$ into Equation (41), we obtain the solution form as Equation (14). This implies that the Case 3 is equivalent to

Table 1. Comparison between Aslan and Ozis (2009) solutions and our solutions.

Aslan and Ozis (2009) solution	Our solution
i. If $\lambda=1,\ \mu=0,\ a_0=\frac{-1}{3},\ c=\mp\sqrt{2}$ and $\xi_0=0,$ Equation (19) becomes $u_{7,8}(x,t)=\frac{-1}{3}+\frac{1}{2}\sec h^2\frac{1}{2}\left(x\pm\sqrt{2}\ t\right).$	i. If $a_1=\frac{-1}{3},$ solution Equations (23) and (37) become $u(x,t)=\frac{-1}{3}+\frac{1}{2}\sec h^2\frac{1}{2}\left(x\pm\sqrt{2}\ t\right).$
ii. If $\lambda^2-4\mu=1$ and $\xi_0=0$ Equation (24) becomes $u_{17,18}(x,t)=\frac{-1}{3}+\sec h^2\frac{1}{2}\left(x\pm\sqrt{2}\ t\right).$	ii. Solution Equations (21) and (33) become $u(x,t)=\frac{-1}{3}+\sec h^2\frac{1}{2}\left(x\pm\sqrt{2}\ t\right).$

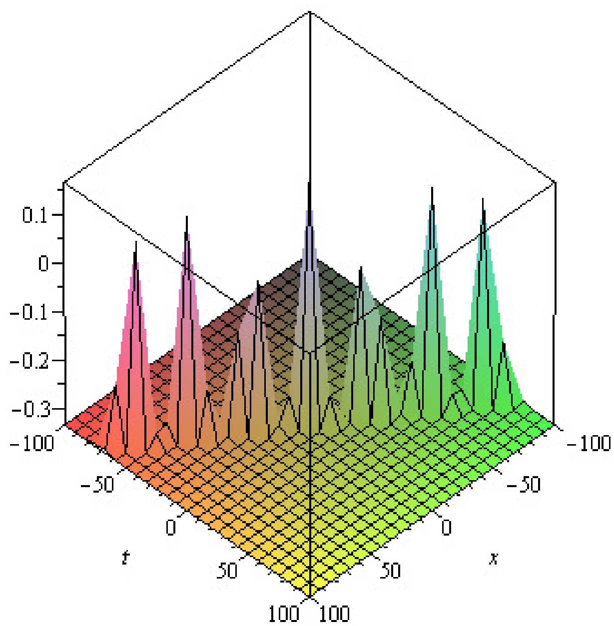

Figure 1. Illustrated solitons of Equations (21) and (33).

to the Case 1.

Also, if we consider $p=c=3$ and $q=d=3,$ it can be shown that this Case is also equivalent to the Cases 1 and 2.

Therefore, we think that no need to find the solutions again. It is noted that the solution Equations (21) and (33) are identical. And, the solution Equation (37) becomes the solution Equation (23) if a_0 is replaced by a_1. Moreover, the solution Equations (25) and (39) are identical.

DISCUSSION

Many authors implemented different methods to the sixth-order Boussinesq equation for obtaining travelling wave solutions, such as, Wazwaz (2008) used the Hirota's bilinear method and the tanh-coth method for constructing multiple-soliton solutions. Hosseini et al. (2011) implemented the exponential rational function method for getting exact traveling wave solutions and Aslan and Ozis (2009) applied the (G'/G)-expansion method to construct analytical solutions. To the best of our awareness, the sixth-order Boussinesq equation (Aslan and Ozis, 2009) has not been investigated by the exp-function method to construct exact travelling wave solutions. The obtained solitary wave solutions are new and have not been found in the previous literature.

Beyond Table 1 Aslan and Ozis (2009) obtained other trigonometric solutions (20) and (25). But, we obtain more new solutions (25), (35) and (39).

Graphical representations of exact solitary wave solutions

Our obtained solutions are shown in Figures 1 to 4 with the aid of Maple 13.

Solutions of the regularized long lave (RLW) equation

The regularized long wave equation is an important nonlinear wave equation. It is related to a huge number of important physical phenomena, such as, shallow water waves, plasma waves and ion acoustic plasma waves. The RLW equation is an alternative depiction of nonlinear dispersive waves to the more usual Korteweg-de Vries (KdV) equation (Soliman, 2005).

Now, let us consider the regularized long wave (RLW) equation (Eilbeck and Guire, 1977):

$$u_t+u_x+uu_x-u_{txx}=0. \tag{42}$$

Now, we seek solitary wave solutions of Equation (42). Using Equation (2), Equation (42) transformed to an ordinary differential equation:

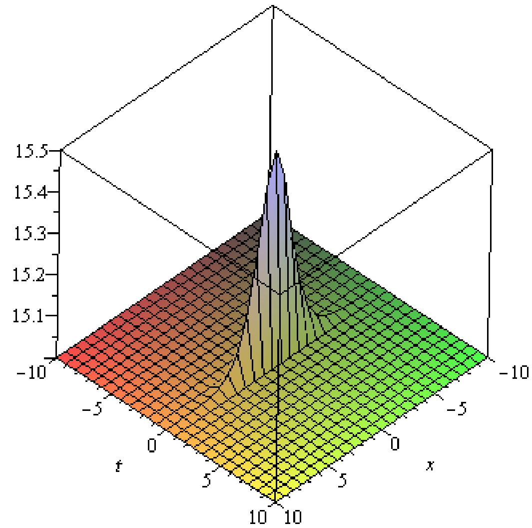

Figure 2. Single solitary wave of Equations (23) and (37).

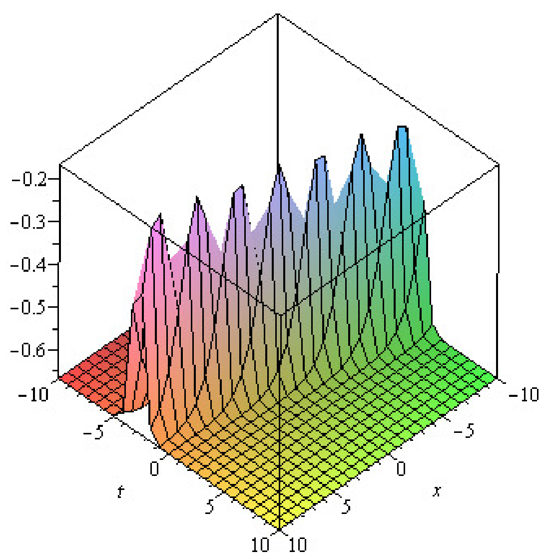

Figure 3. Solitons of Equations (25) and (39).

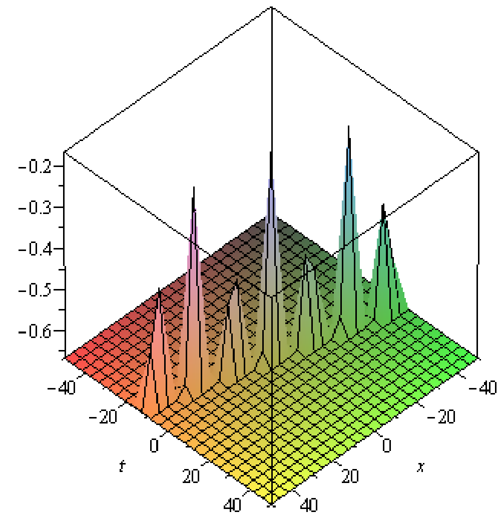

Figure 4. Solitons solutions of Equations (25) and (39).

$$su' + u' + uu' - su''' = 0. \tag{43}$$

where primes denote the derivative with respect to ξ. According to He and Wu (2006), the solution of Equation (43) can be expressed in the form of Equation (5).

In order to determine the values of c and p, we balance the highest order linear term of u''' with the highest order nonlinear term of $u\,u'$ in Equation (43). Therefore, we have

$$u''' = \frac{c_1 \exp\left[(3p+c)\xi\right] + \ldots}{c_2 \exp\left[4p\xi\right] + \ldots}, \tag{44}$$

and

$$uu' = \frac{c_3 \exp\left[2(p+c)\xi\right] + \ldots}{c_4 \exp\left[4p\xi\right] + \ldots}, \tag{45}$$

Balancing the highest order of the exp-function, in Equations (44) and (45), we obtain $3p + c = 2(p+c)$, which in turn gives $p = c$.

To determine the values of d and q, we balance the lowest order linear term of u''' with the lowest order nonlinear term of uu' in Equation (43). Therefore, we have

$$u''' = \frac{\ldots + d_1 \exp\left[-(d+3q)\xi\right]}{\ldots + d_2 \exp\left[-4q\xi\right]}, \tag{46}$$

and

$$uu' = \frac{\ldots + d_3 \exp\left[-2(d+q)\xi\right]}{\ldots + d_4 \exp\left[-4q\xi\right]}, \tag{47}$$

Now balancing the lowest order of the exp-function, in Equations (46) and (47), we obtain $-(d+3q) = -2(d+q)$, which in turn gives $q = d$.

Case 1

Choose $p = c = 1$ and $q = d = 1$.

For this case, the trial solution Equation (5) reduces to Equation (14). Now, substituting Equation (14) into Equation (43) and simplifying, we obtain

$$\frac{1}{A}\left(C_3e^{3\xi}+C_2e^{2\xi}+C_1e^{\xi}+C_0+C_{-1}e^{-\xi}+C_{-2}e^{-2\xi}+C_{-3}e^{-3\xi}\right)=0. \quad (48)$$

Setting each coefficient of $\exp(\pm n\xi)$, $n=0,1,2,3,...,$ we obtain a set of algebraic equations (which are not shown here with). And, the coefficients of $\exp(\pm n\xi)$, $n=0,1,2,3,...$ to be zero, such as:

$$C_3=0,\ C_2=0,\ C_1=0,\ C_0=0,\ C_{-1}=0,\ C_{-2}=0,\ C_{-3}=0 \quad (49)$$

Solving the system of algebraic Equation (49) with the aid of algebraic software Maple 13, we obtain

$$b_0=b_0,\quad a_0=a_0,\quad a_{-1}=\frac{-1}{4}b_0^2,\quad a_1=-1,\quad b_{-1}=\frac{1}{4}b_0^2,\quad s=\frac{-(a_0+b_0)}{6b_0},$$
(50)

where a_0 and b_0 are arbitrary constants.

Now, substituting Equation (50) into Equation (14) and simplifying, we obtain

$$u(\xi)=-1+\frac{4(a_0+b_0)}{(4+b_0^2)\cosh\xi+(4-b_0^2)\sinh\xi+4b_0}, \quad (51)$$

where $\xi=x-\frac{a_0+b_0}{6b_0}t.$

If $b_0=2,$ Equation (51) becomes

$$u(x,t)=-1+\frac{a_0+2}{2\left[1+\cosh\left(x-\frac{a_0+2}{12}t\right)\right]}. \quad (52)$$

Case 2

Choose $p=c=2$ and $q=d=1.$

For this case, the trial solution Equation (5) reduces to Equation (27).

According to the same procedure as described in case 1, we obtain

$$a_1=a_1,\quad b_1=b_1,\quad a_{-1}=0,\quad a_0=\frac{-1}{4}b_1^2,\quad a_2=-1,\quad b_0=\frac{1}{4}b_1^2,\quad s=\frac{-(a_1+b_1)}{6b_1}, \quad (53)$$

where a_1 and b_1 are arbitrary constants.

Substituting Equation (53) into Equation (27) and simplifying, we obtain

$$u(\xi)=-1+\frac{4(a_1+b_1)}{(4+b_1^2)\cosh\xi+(4-b_1^2)\sinh\xi+4b_1}, \quad (54)$$

where $\xi=x-\frac{a_1+b_1}{6b_1}t.$

If $b_1=2,$ Eq. (54) becomes

$$u(x,t)=-1+\frac{a_1+2}{2\left[1+\cosh\left(x-\frac{a_1+2}{12}t\right)\right]}. \quad (55)$$

Case 3

Choose $p=c=2$ and $q=d=2.$

For this case, the trial solution Equation (5) reduces to Equation (40).

For simplicity, we may consider $b_2=1,$ $b_{-2}=0$ and $b_{-1}=0.$ Then the solution Equation (40) is simplified:

$$u(\xi)=\frac{a_2e^{2\xi}+a_1e^{\xi}+a_0+a_{-1}e^{-\xi}+a_{-2}e^{-2\xi}}{e^{2\xi}+b_1e^{\xi}+b_0}. \quad (56)$$

According to same procedure as described in case 1, we obtain

$$a_1=a_1,\quad b_1=b_1,\quad a_{-2}=0,\quad a_{-1}=0,\quad a_0=\frac{-1}{4}b_1^2,\quad a_2=-1,\quad b_0=\frac{1}{4}b_1^2,\quad s=\frac{-(a_1+b_1)}{6b_1},$$
(57)

where a_1 and b_1 are arbitrary constants.

Using Equation (57) into Equation (56) and simplifying, we obtain

$$u(\xi)=-1+\frac{4(a_1+b_1)}{(4+b_1^2)\cosh\xi+(4-b_1^2)\sinh\xi+4b_1}, \quad (58)$$

where $\xi=x-\frac{a_1+b_1}{6b_1}t.$

If $b_1=2,$ Equation (58) becomes

$$u(x,t)=-1+\frac{a_1+2}{2\left[1+\cosh\left(x-\frac{a_1+2}{12}t\right)\right]}. \quad (59)$$

It is noted that the solution Equations (55) and (59) become the solution Eq. (52), if a_1 is replaced by a_0.

Many researchers used different methods to find traveling wave solutions for RLW equation, such as; Eilbeck and Guire (1977) used the numerical method, and El-Danaf et al. (2005) investigated the Adomian

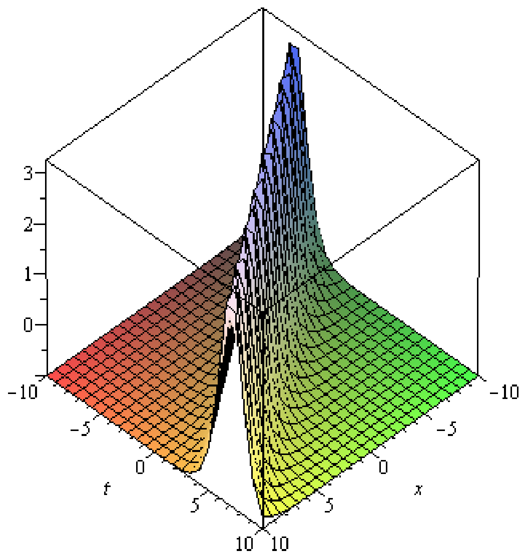

Figure 5. Soliton solution of Equations (49), (52) and (56).

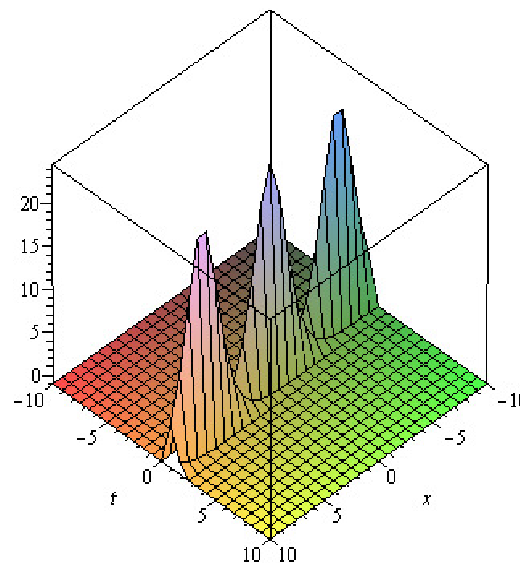

Figure 7. Solitons solutions of Equations (49), (52) and (56).

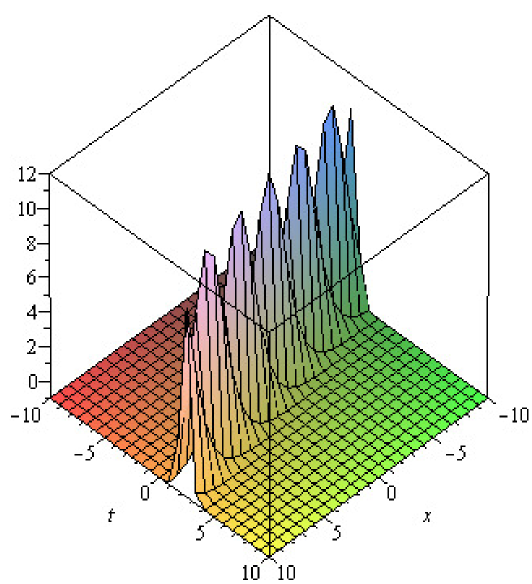

Figure 6. Solitons of Equations (49), (52) and (56).

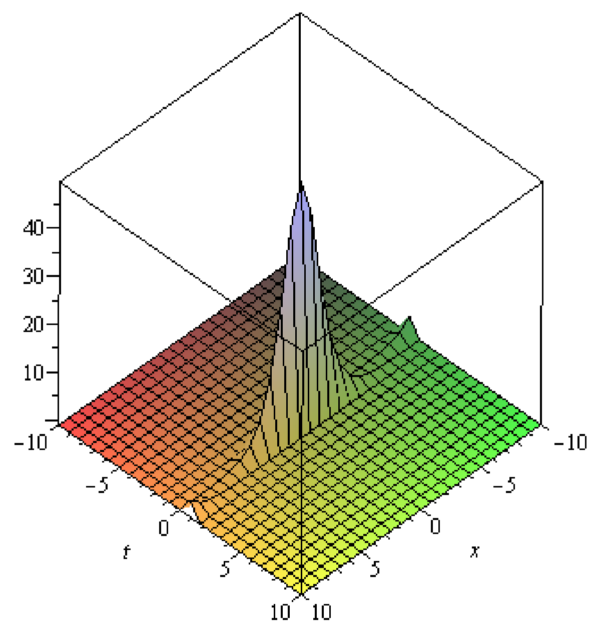

Figure 8. Single soliton of Equations (49), (52) and (56).

decomposition method. But, to the best of our knowledge, no body studied the equation by using the exp-function method for searching exact traveling wave solutions. Obtained solitary wave solutions in this article are new, which have not been found in the previous literature.

Graphical representations of solitary wave solutions

The aforementioned solutions are presented in Figures 5 to 8 with the help of Maple 13.

Conclusions

In this article, we obtain new exact traveling wave solutions including solitary solutions for the Sixth-order Boussinesq equation and RLW equations by using the exp-function method. The obtained solutions show that the exp-function method is promising and powerful mathematical tool for solving nonlinear evolution equations which arise in mathematical physics, engineering sciences and applied mathematics. We hope that the method can be effectively used for further studies to many NLEEs.

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REFERENCES

Abdou MA, Soliman AA, El-Basyony ST (2007). New application of Exp-function method for improved Boussinesq equation. Phy. Lett. A, 369: 469-475.

Ablowitz MJ, Clarkson PA (1991). Solitons, nonlinear Evolution Equations and Inverse Scattering Transform. Cambridge Univ. Press, Camb., 561p.

Asgari A, Bagheripour MH, Mollazadeh M (2011). A generalized analytical solution for a nonlinear infiltration equation using the exp-function method. Scientia, Iranica A, 18(1): 28-35.

Aslan I, Ozis T (2009). On the validity and reliability of the (G'/G)-expansion method by using higher-order nonlinear equations. Appl. Math. Comput., 211: 531-536.

Bekir A, Aksoy E (2010). Exact solutions of nonlinear evolution equations with variable coefficients using exp-function method. Appl. Math. Comput., 217: 430-436.

Eilbeck JC, Guire GRM (1977). Numerical study of the regularized long-wave equation. II: Interaction of solitary waves. J. Comput. Phys., 23: 63-73.

El-Danaf TS, Ramadan MA, Alaal FEIA (2005). The use of Adomian decomposition method for solving the regularized long-wave equation. Chaos, Soli. Frac., 26: 747-757.

Ezzati R, Aqhamohamadi M (2011). Using homotopy analysis method to obtain approximate analytical solutions of wave equations. Int. J. Phys. Sci., 6(15): 3821-3826.

He JH (1997). Variational iteration method for delay differential equations. Commun. Nonlinear Sci. Numer. Simulat., 2 (4): 235-236.

He JH, Abdou MA (2007). New periodic solutions for nonlinear evolution equations using Exp-function method. Chaos, Soli. Frac., 34: 1421-1429.

He JH, Wu XH (2006). Exp-function method for nonlinear wave equations. Chaos, Soli. Frac., 30: 700-708.

Hirota R (1971). Exact solution of the KdV equation for multiple collisions of solutions. Phys. Rev. Lett., 27: 1192-1194.

Hosseini M, Abdollahzadeh H, Abdollahzadeh M (2011). Exact travelling solutions for the sixth-order Boussinesq equation. J. Math. Comp. Sci., 2(2): 376-387.

Jafari M, Hosseini MM, Mohyud-Din ST, Ghovatmand M (2011). Solution of the nonlinear PDAEs by variational iteration method and its applications in nanoelectronics. Int. J. Phy. Sci., 6(6): 1535-1539.

Liao SJ (1992). The Homotopy Analysis Method and its applications in mechanics. Ph. D. Dissertation (in English), Shanghai Jiao Tong Univ., http://numericaltank.sjtu.edu.cn/

Liu S, Fu Z, Liu S, Zhao Q (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. Phys. Lett. A, 289: 69-74.

Malfliet W (1992). Solitary wave solutions of nonlinear wave equations. Am. J. Phys., 60: 650-654.

Misirli E, Gurefe Y (2011). Exp-function method for solving nonlinear evolution equations. Math. Comput. Appli., 16(1): 258-266.

Mohyud-Din ST, Yildirim A, Usman M (2011). Homotopy analysis method for fractional partial differential equations. Int. J. Phys. Sci., 6(1): 135-145.

Naher H, Abdullah F, Akbar MM (2011). New traveling wave solutions of the higher dimensional nonlinear partial differential equation by the Exp-function method. J. Appl. Math., (Article ID: 575387, in press).

Neirameh A, Ghasemi R, Roozi A (2010). Traveling wave solution for non-linear Klein-Gordon equation. Int. J. Phys. Sci., 5(16): 2528-2531.

Rogers C, Shadwick WF (1982). Backlund Transformations, Academic Press, New York. 352p.

Soliman AA (2005). Numerical simulation of the generalized regularized long wave equation by He's variational iteration method. Math. Comp. Simu., 70: 119-124.

Wazwaz AM (2008). Multiple-soliton solutions for the ninth-order KdV equation and sixth-order Boussinesq equation. Appl. Math. Comput., 203: 277-283.

Wu XH, He JH (2007). Solitary solutions, periodic solutions and compacton-like solutions using the Exp-function method. Comp. Math. Appli., 54: 966-986.

Yildirim A, Pinar Z (2010). Application of the exp-function method for solving nonlinear reaction-diffusion equations arising in mathematical biology. Comp. Math. Appl., 60: 1873-1880.

Yousefi SA, Lotfi A, Dehghan M (2009). He's variational iteration method for solving nonlinear mixed Voltera-Fredholm integral equations. Comp. Math. Appl., 58: 2172-2176.

Zhang JF, Dai CQ, Yang Q, Zhu JM (2005). Variable-coefficient F-expansion method and its application to nonlinear Schrodinger equation. Optics Commun., 252: 408-421.

Zhang S (2010). Exp-function method for Riccati equation and new exact solutions with two arbitrary functions of $(2+1)-$ dimensional Konopelchenko-Dubrovsky equations. Appl. Math. Comput., 216: 1546-1552.

Zhang S (2008). Exp-function method for constructing explicit and exact solutions of a lattice equation. Appl. Math. Comput., 199: 242-249.

Zhou YB, Wang ML, Wang YM (2003). Periodic wave solutions to coupled KdV equations with variable coefficients. Phys. Lett. A, 308: 31-36.